Turku PET Centre Modelling report TPCMOD0008 2003-01-19
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Noise model for PET time-radioactivity curves

This document reviews the noise model that is used to simulate the noise (variance) in PET measurements.

Noise model

Time-radioactivity concentration curve measured from region drawn on reconstructed PET image consists of independent numbers of counts averaged over the scan time of a time frame. Thus the measurement noise variance is proportional to the imaged radioactivity concentration and is inversely proportional to the scan duration. Therefore the variance of the measurement error [Chen et al. 1991] can be described as:

$$\sigma^2(t'_k) = \frac{c \times y(t'_k)}{\Delta t_k} \qquad (1)$$

where $\sigma^2(t'_k)$ is the variance of the k^{th} measurement of radioactivity concentration y (not corrected for decay) at frame mid-time t'_k, Δt is the frame length, and the coefficient c is a proportionality constant which actually determines the noise level in the measurement. This noise model is equivalent to the experimental PET variance model used in [Jovkar et al. 1989].

The variance in equation (1) is dependent on the measured counts, and thus on the concentration of radioactive label. In PET modelling, the interest is in concentration of labelled tracers, which are proportional to the decay corrected radioactivities. Thus the measured and simulated PET data is usually corrected for decay to the tracer injection time. The non-decay corrected radioactivity concentration in eq. (1) can be replaced with decay corrected radioactivity concentration $ROI(t'_k)$:

$$\sigma^2(t'_k) = \frac{c \times ROI(t'_k) \times e^{-\lambda t'_k}}{\Delta t_k} \qquad (2)$$

where λ is the decay constant of the isotope, which can be computed from the isotope half-life, $T_{½}$, from equation $\lambda = ln\ 2/T_{½}$. Here $ROI(t'_k)$ represents the noise-free concentration, not the measured concentration which already contains an unknown error. The standard deviation of the measurement can be calculated by taking a square root of the variance. The standard deviation can be decay corrected by multiplying the equation with decay term $e^{\lambda t'k}$, and thus the equation (3) is be derived to calculate the standard deviation ($SD(t'_k)$) for decay corrected radioactivity concentration:

$$SD(t'_k) = \sqrt{\frac{c \times ROI(t'_k) \times e^{\lambda t'_k}}{\Delta t_k}} \qquad (3)$$

Note that the sign in the decay term inside the square root was changed while the decay term outside the square root was cancelled out in the eq. (3).

Noise can be added to a simulated, error-free, PET time-radioactivity concentration curve using eq. (4):

$$ROI_N(t'_k) = ROI(t'_k) + SD(t'_k) \times G(0,1) \qquad (4)$$

where *G(0,1)* is a pseudo-random number from Gaussian distribution with zero mean and standard deviation of one.

Apart from notational differences, the equations are equivalent to the ones used by Feng *et al.* (1991), Logan *et al.* (2001) and Varga & Szabo (2002), except that Logan *et al.* replaced the proportionality constant c with a scale factor (*Sc*), which was placed outside of the square root; thus, $c=\sqrt{Sc}$. For the examples presented by Logan et al. (2001), *Sc* ranged from 0.25 to 8; corresponding range of proportionality constant is from 0.5 to 2.8.

Noise distribution

The noise model here is random, related to counting statistics. In case of pure radioactivity counting over some time interval, measured values have a Poisson distribution and variance can be calculated as a square root of count number. However, this is not adequate for reconstructed PET image data [Budinger et al. 1978]. Because the variances are not known, uniform variance and/or white noise have been used in some instances [Coxson et al. 1997]. Since there are a number of sources of noise in the PET image, and several additive sources of errors tend to form a Gaussian distribution, Gaussian distribution with zero mean is assumed here.

Studies which have applied this noise model

Logan et al. (2001) used this noise model to simulate the bias caused to Logan plot slope by the measurement noise and to validate the method for removing the bias. For exactly same purpose, the same noise model was used by Varga and Szabo (2002). Naturally, the noise model is extensively used by the group of Chen and Feng. The weighting methods applied in estimation of parameters of non-linear compartmental models are usually based on this noise model.

Other noise models

In the early days of PET imaging, an empirically derived equation method [Budinger et al. 1977] was widely applied in simulation studies. Ikoma et al. have developed a dynamic digital phantom, which contains a related noise model [Ikoma et al. 1998].

Generating Gaussian pseudo-random numbers

Assuming that we have a source of uniform pseudo-random numbers in the range from 0 to 1, the pseudo-random numbers which have a Gaussian (normal) distribution with zero mean and a standard deviation of one can be generated using e.g. Box-Müller transformation [Box & Muller, 1958]. Below is given a C code for computing the polar form of the Box-Müller transformation:

```
double gaussdev()
{
  static int ready=0, first=1;
  static double dev;
  double fac, rsq, a, b;
  if(first) {first=0; srand(893165470L);}

  /* If we don't have deviate already, then we'll have to make one */
  if(!ready) {
    do {
      a=2.*(double)rand()/(double)RAND_MAX - 1.0;
      b=2.*(double)rand()/(double)RAND_MAX - 1.0;
      rsq=a*a + b*b;
    } while (rsq>=1.0 || rsq==0.0);
    fac=sqrt(-2.0*log(rsq)/rsq);
    dev=a*fac; ready=1;
    return(b*fac);
  } else { /* Deviate is ready, just return it */
    ready=0;
    return(dev);
  }
}
```

Since the same seed number is used on the first time that this routine is called, the same set of random deviates will be returned to the calling program each time it is run. If different set of pseudo-random numbers is required, the seed can be set from system clock; however, then the program should not be run repeatedly during one second.

References

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